Modeling of the seismic behavior of shear-critical reinforced concrete columns

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Inelastic failure of reinforced concrete (RC) structures under seismic loadings can be due either to loss of flexural, shear, or bond capacity. This paper describes the formulation of an inelastic nonlinear beam element with axial, bending, and shear force interaction. The element considers shear deformation and is based on the section discretization into fibers with hysteretic models for the constituent materials. The steel material constitutive law follows the Menegotto–Pinto model. The concrete model is based on a smeared approach of cracked continuous orthotropic concrete with the inclusion of Poisson effect. The concrete model accounts for the biaxial state of stress in the directions of orthotropy in accordance with the Softened Membrane Model, in addition to degradation under reversed cyclic loading. The shear mechanism along the beam is modeled using a Timoshenko beam approach. Transverse strains are internal variables determined by imposing equilibrium between concrete and transverse reinforcements. Element forces are obtained by performing equilibrium based numerical integration on section axial, flexural and shear behaviors along the length of the element. Dynamic behavior was accounted for by adopting the well-known Newmark approach. Rayleigh damping was assumed to simulate the damped behavior under seismic excitations. In order to establish the validity of the proposed model correlation studies were conducted between analytical results and experimental data of RC columns tested under the shake table.

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1. Introduction

The seismic behavior of RC structures is strongly affected by the earthquake intensity as well as the combined effect of bending, shear, and axial force interaction. The establishment of accurate constitutive models of RC elements for seismic analysis under combined loadings is essential to predicting the correct behavior of the structure. The past three decades witnessed the development of new constitutive models in an effort to improve the general numerical performance of RC structures. In these models, the equilibrium equations assume the stresses in the concrete struts and steel bars to be smeared. Similarly, the strains of steel and concrete are also smeared, and are obtained by averaging the strains along a steel bar that crosses several cracks. The constitutive laws of concrete and steel bars were developed through large-scale panel testing, and relate the smeared stresses to the smeared strains of the element [1–3]. The first work to develop such constitutive laws was the one by Vecchio and Collins [4], who proposed the Compression Field Theory (CFT) to predict the nonlinear behavior of cracked reinforced concrete membrane elements. The CFT however was unable to take into account the tension stiffening effect of the concrete. The researchers later improved their model and developed the Modified Compression Field Theory or MCFT [5], in which the tension stiffening of concrete was accounted for by imposing a concrete tensile stress across the shear crack. Belarbi and Hsu [1,2], and Pang and Hsu [6] used a different approach and developed the Rotating-Angle Softened-Truss Model (RA-STM). In this model, the tension stiffening effect of concrete was taken into account by assuming a shear stress along the crack direction. Later, the researchers improved their work and developed the Fixed-Angle Softened-Truss Model (FA-STM) [7–9], which is capable of predicting the concrete contribution to shear resistance by assuming the cracks to be oriented at a fixed angle. Recently, Zhu et al. [10] derived a rational shear modulus and developed a simple solution algorithm for the FA-STM. The work was further extended by developing the Hsu/Zhu Poisson ratios [11], which led to the development of the Softened Membrane Model (SMM), which can accurately predict the entire behavior of the specimen, including both the pre- and post-peak responses. The SMM model is very appealing for modeling the shear behavior of concrete elements as it is based on solid mechanics of materials fundamentals and is accurately calibrated from panel testing. The only attempt to implement the SMM model into finite element codes was conducted by Mo et al. [12], who used it to develop a plane stress element for RC shear walls subjected to cyclic loading conditions. Their model was...
1–2 direction of applied principal tensile stress
$x$–$y$ local coordinate of RC element
$x$–$y'$ local coordinate of RC element for circular cross section
$\theta'$ angle for circular cross section
$r$–$d$ concrete principal coordinate system, $r_x$ force interpolation function
$S(x)$ section forces at a section $x$
$s(x)$ section deformations at a section $x$
$s'$ section deformations at $x$–$y'$ coordinate system
$p$ element end forces
d element deformation
$K_{\text{Section}}(x)$ section stiffness matrix
$r(x)$ residual section deformation
$r_d$ residual of sectional deformation
$r_u$ element residual deformation vector
$f(x)$ element flexibility matrix without rigid body modes
$\alpha_m$, $\beta_k$ Rayleigh damping coefficients
$\Delta t$ time increment
$\Delta F$ dynamic residual load increment
$k$, $k_0$ dynamic stiffness
$k_d$ static stiffness
$d$ velocity
$d_0$ acceleration
$V_c$ shear strength provided by the concrete
$V_s$ shear strength provided by the transverse reinforcement
$\alpha_1$ angle between the $(x$–$y)$ coordinate system and $(1$–$2)$ coordinate system
$\alpha_r$ angle between the $(x$–$y)$ coordinate system and $(r$–$d)$ coordinate system
$\varepsilon_x$ smeared biaxial strain in the x-direction
$\varepsilon_y$ smeared biaxial strain in the y-direction
$\varepsilon_0$ concrete cylinder strain corresponding to peak cylinder strength $f'_c$
$\varepsilon_d$ strain in the reinforcement that yield first
$\varepsilon_{1x}$ equivalent uniaxial principal strain in the reinforcement in x-direction
$\varepsilon_{1y}$ equivalent uniaxial principal strain in the reinforcement in y-direction
$\varepsilon_{1p}$ ultimate strain in the 1-direction
$\varepsilon_{2p}$ ultimate strain in the 2-direction
$\varepsilon_y$ strain perpendicular to the stirrup cross section
$\varepsilon_1$ smeared biaxial strain in the 1-direction
$\varepsilon_1$ smeared uniaxial strain in the 1-direction
$\varepsilon_2$ smeared biaxial strain in the 2-direction
$\varepsilon_2$ smeared uniaxial strain in the 2-direction
$\gamma_{12}$ smeared shear strain in the 1–2 coordinate system
$\gamma_{xy}$ smeared shear strain in the x–y coordinate system
$\sigma_x$, $\sigma_y$ applied normal stress in the x-direction
$\tau_{xy}$ applied shear stress in the x-y coordinate system
$\sigma_{1x}$ concrete stress in the x-direction of fiber $i$
$\sigma_{1y}$ concrete stress in the y-direction of fiber $i$
$\sigma_{1p}$ ultimate stress in the 1-direction
$\sigma_{2p}$ ultimate stress in the 2-direction
$\sigma_1$ smeared stress in the 1-direction
$\sigma_2$ smeared stress in the 2-direction
$\sigma_1'$ smeared stress in the 1-direction
$\sigma_2'$ smeared stress in the 2-direction
$\sigma_{12}'$ concrete smeared shear stress in the 1–2 coordinate system
$\sigma_{12}$ reinforcement ratio of the steel layer in the $i$th direction
$\rho_w$ wall reinforcement ratio
$f_{sx}$ reinforcing bar stresses along the $x$ directions respectively
$f_{sy}$ reinforcing bar stresses along the $y$ directions respectively
$\rho_{sx}$ smeared steel ratio in the direction of $x$
$\rho_{sy}$ smeared steel ratio in the direction of $y$
$A_{i,y}$ area of concrete between the spacing of the stirrups in the $y$ direction of fiber $i$
$A_{i,x}$ area of steel between the spacing of the stirrups in the $x$ direction of fiber $i$
$A_{cx}$ area of the concrete fiber in the $x$ direction.
$\mu_{12}$ Hsu/Zhu ratio (effect of strain in 2-direction on strain in 1-direction)
$\mu_{21}$ Hsu/Zhu ratio (effect of strain in 1-direction on strain in 2-direction)
$\alpha_{i1}$ deviation angle between the applied stress angle $\alpha_1$ and the rotating angle $\alpha_i$
$K_{i1}$, $K_{i2}$ biaxial strength magnification factors in the $1$–$2$ direction
$E_{1x}$ concrete tangent uniaxial modulus in the $1$-direction
$E_{2x}$ concrete tangent uniaxial modulus in the $2$-direction
$E_{sx}$ uniaxial steel stiffness from the Menegotto–Pinto model along the $x$-direction
$E_{sy}$ uniaxial steel stiffness from the Menegotto–Pinto model along the $y$-direction
$G_{12}$ concrete shear modulus
$R$ rotation matrix
$F_x$ section forces in the longitudinal direction
$F_y$ section forces in the transverse direction
$M_2$ section forces in the rotational direction
$[\mu]$ Hsu/Zhu matrix
$[A]$ transformation matrix
$[\varepsilon_{12}]$ local principal strain vector
$[m]$ mass matrix
$[c]$ Rayleigh damping matrix
$[\sigma_{1x}]$ the local concrete stress vector
$[D_{0x}]$ local uniaxial concrete material secant stiffness matrix in the principal direction
$[D_p]$ concrete orthotropic stiffness matrix in the global $x$–$y$ direction
$[D_{0x}]^x$ longitudinal steel local stiffness matrix
$[D_{0y}]^y$ transverse steel local stiffness matrix,
$[D_p]^x$ reinforcement stiffness matrix in the global direction
$[D_p]^y$ longitudinal steel global stiffness matrix
$[D_p]_{xy}$ transverse steel global stiffness matrix
$[D_p]^{x+y}$ stiffness matrix including concrete and transverse steel terms
$[k_{\text{fiber}}]$ condensed fiber stiffness
$[\sigma_{\text{fiber}}]$ condensed concrete fiber stresses
$[T]$ transformation matrix
$[(K_{\text{Section}})_{xx}]$ contribution of concrete to the section stiffness
$[(K_{\text{Section}})_{xx}]$ sectional stiffness due to the longitudinal reinforcement
$[(F_{\text{Section}})_{xx}]$ sectional forces due to the concrete fiber
$[(F_{\text{Section}})_{xx}]$ sectional forces due to the longitudinal steel fiber
$[K_{\text{Section}}]$ total stiffness of the section
$[F_{\text{Section}}]$ total force of the section
$[F_{\text{Element}}]$ element force vector
$[K_{\text{Element}}]$ element stiffness matrix.
validated through a correlation study with experimentally tested framed shear wall systems. Excellent results were obtained when compared to experimental data.

Modeling of the shear behavior of reinforced concrete structures is typically performed with two-dimensional continuum elements. Such models can accurately describe the local behavior of the element. Continuum models are computationally very expensive though, which limits their applicability to simulate the behavior of large structural systems. Beam–column elements, on the other hand, have proved to be able to model the flexural behavior of concrete structures rather well, and are computationally very efficient. With the inclusion of shear deformations, these elements can also accurately simulate the behavior of reinforced concrete columns dominated by shear as well as flexural behavior. Earlier beam–column elements were based on plastic hinge formulations. Shear springs to account for shear effects were developed to be used in parallel with plastic hinge elements (e.g. D’Ambri [13], Li and Jirsa [14], Ricles et al. [15], and Pincheira et al. [16]). The shear–axial-bending interaction in these models, though, is not accounted for at the section level. Recently, Elwood [17], Lee and Elnashai [18,19], and Zhang and Xu [20] developed a spring element with axial-shear interaction to be used in series with beam–column elements.

Beam–column elements with distributed inelasticity have the advantage of predicting the spread of plasticity along the element length, and are therefore considered more accurate than lumped plastic hinge elements. In particular, fiber-based distributed beam elements have the ability to describe the inelastic behavior along the depth by discretizing the section into several fibers with appropriate constitutive behavior. Spacone et al. [21] developed a force-based fiber model for the analysis of frame elements. Their model accounted for the axial–flexural interaction effect. Ranzo and Petrangeli [22], and Shirai et al. [23] proposed a shear element in series with force-based fiber models. Marini and Spacone [24] developed a fiber-based beam element that accounts for shear effects. The shear behavior was uncoupled, however, from the flexural behavior. Petrangeli et al. [25] extended force-based fiber elements to account for shear–axial–flexure interaction effects using a fiber constitutive material model based on the microplane approach. Mazars et al. [26] followed the same approach and developed a multi-fiber Timoshenko beam element with the concrete constitutive law proposed by La Borderie [27] and based on damage mechanics. These mechanics-based models however did not consider the dynamic behavior of reinforced concrete columns subjected to seismic excitations.

The objective of the paper is to evaluate the seismic behavior of concrete bridge columns using a fiber beam–column element formulation. A Timoshenko beam theory was adopted in the model to account for shear deformation effects. To predict the entire pre- and post-peak responses of the shear stress–strain curves, the SMM model was adopted and the Hsu/Zhu ratios (Poisson ratios of cracked reinforced concrete) were taken into effect. The study focused on the performance of RC bridge columns subjected to real time earthquake excitations. During an earthquake, RC structures undergo large cyclic deformations with the concrete undergoing crack opening and closing. The problem is further complicated by considering softening, confinement, tension stiffening and dilatancy, which accompanies large shear strains induced in the transverse reinforcement. This study tackles the extension of the fiber-based beam element formulation to account for axial, bending and shear interaction. Such interaction is essential to accurately predict the complex behavior of RC members under combined seismic loads. The model was developed using the finite element program FEAPpv [28]. The fiber beam–column formulation for shear-critical elements is presented first.

### 2. Finite element formulation

The finite element model is based on the recently developed force method of analysis [21,29,30]. The force-based formulation proved to overcome most of the difficulties associated with the standard displacement approach and generally has superior robustness and requires a lower number of model degrees of freedom for comparable accuracy when dealing with RC members.

The generalized nodal forces and nodal deformations of the element are shown in Fig. 1, where the rigid body modes are excluded. Thus the element force vector and element deformation vector respectively are:

\[
P = \begin{bmatrix} N \\ M \end{bmatrix}, \quad \delta = \begin{bmatrix} u \\ \beta_1 \\ \beta_2 \end{bmatrix}.
\]

(1)

The section forces and corresponding section deformations are:

\[
S(x) = \begin{bmatrix} N(x) \\ V(x) \\ M(x) \end{bmatrix}, \quad s(x) = \begin{bmatrix} \varepsilon(x) \\ \gamma(x) \\ \chi(x) \end{bmatrix}.
\]

By utilizing the force interpolation function \( b(x) \), the section forces \( S(x) \) at a section \( x \) are related to the element end forces \( P \) by:

\[
S(x) = b(x)P
\]

(3)

where

\[
b(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & \frac{x}{L} & -1 \end{bmatrix}.
\]

(4)

Element-wise compatibility is enforced through the application of the virtual force principle:

\[
\delta P^T \delta d = \int_0^L \delta S^T(x) \delta s^T(x) dx.
\]

(5)

The Section force vector \( S(x) \) and the section stiffness matrix \( K_{\text{section}}(x) \) are determined from a known section deformation vector \( s(x) \). The section deformation increments \( \Delta s(x) \) are determined by adding the residual section deformation \( r(x) \) from the previous iteration to the deformation caused by the section force increments \( \Delta S(x) \):

\[
\Delta s(x) = f(x) \Delta S(x) + r(x)
\]

(6)

where \( f(x) = [K_{\text{section}}(x)]^{-1} \) is the section flexibility matrix.

After substituting Eqs. (3) and (6) in Eq. (5) and eliminating \( \delta P^T \) from the resulting equation due to its arbitrariness, the element compatibility equation is written as:

\[
\Delta d = \bar{F} \Delta P + r_d
\]

(7)

where \( \bar{F} \) is the element flexibility matrix without rigid body modes, and is evaluated as follows:

\[
\bar{F} = \int_0^L b^T(x)f(x)b(x)dx
\]

(8)
and $r_a$ is the element residual deformation vector:

$$r_a = \int_0^L b^T(x)r_d(x)dx$$

(9)

whereas $r_d$ is the residual of sectional deformations.

To implement the force-based model in a finite element program based on displacement degrees of freedom, Eq. (7) needs to be inverted:

$$K_{\text{element}} \Delta d = R_u$$

(10)

where the element stiffness matrix $K = \hat{F}^{-1}$, and the resisting load increment $R_u = \Delta P + \hat{F}^{-1} \Delta u$, $\hat{F}^{-1}$ being evaluated from the previous iteration. The process of the state determination of the force-based element requires internal element iteration in addition to the Newton–Raphson global iteration, and is duly described in [21,29].

As stated earlier, the section behavior is evaluated through fiber discretization with appropriate material constitutive models. The constitutive behavior of the concrete and steel fibers is described in subsequent sections. To account for dynamic effects, the Newmark method is adopted and is briefly described in the next section.

3. Dynamic behavior formulation

The nonlinear dynamic behavior is analyzed following the well known Newmark method [31]. Accordingly, the following equilibrium equation is solved for:

$$k_j \dot{\Delta}d_j = \Delta \ddot{F}_j$$

(11)

where $\Delta d_j$ is the displacement increment at time step $j$, and the dynamic stiffness $\hat{k}_j$ is evaluated as follows:

$$\hat{k}_j = k_j + \frac{\gamma}{\beta \Delta t} c + \frac{1}{\beta (\Delta t)^2} m$$

(12)

where $k_j$ is the static stiffness, $m$ is the mass matrix, $c$ is the Rayleigh damping matrix, and $\Delta t$ represents the time increment from steps $j$ to $j + 1$. In the current model the coefficients $\gamma = 0.5$ and $\beta = 0.25$ were selected for considering the average acceleration method, which is unconditionally stable.

The dynamic residual load increment $\Delta \ddot{F}_j$ is evaluated as follows:

$$\Delta \ddot{F}_j = \Delta F_j + \left(\frac{1}{\beta \Delta t} m + \frac{\gamma}{\beta} c\right) \dot{d}_j + \left[\frac{1}{2\beta} m + \Delta t \left(\frac{\gamma}{2\beta} - 1\right) c\right] \ddot{d}_j$$

(13)

where $\dot{d}_j$ and $\ddot{d}_j$ are the velocity and acceleration respectively at time step $j$.

The damping matrix is determined based on the Rayleigh-type formulation expressed as a linear combination of the mass matrix $m$ and the stiffness matrix $k$ [32] as follows:

$$c = \alpha_m m + \beta_k k$$

(14)

where the parameters $\alpha_m$ and $\beta_k$ are calculated based on the natural frequencies of the first modes and their damping ratios.

For a nonlinear system, the incremental procedure described in Eq. (11) needs to be accomplished using an iterative procedure.

4. Concrete constitutive model

According to the ACI 318 building code [33], the ultimate strength of an RC beam is the combination of shear strength provided by the concrete ($V_c$) and shear strength provided by the transverse reinforcement ($V_r$). The term $V_c$ cannot be estimated in the aforementioned CFT, MCFT and RA-STM theories, because the crack angles in these models are assumed to be rotating. The recent FA-STM and SMM theories are capable of accurately accounting for the effect of $V_c$ because the crack angle is based on the principal coordinates of the applied stresses of the RC element with due consideration to the concrete shear stress term ($\tau_c$). The RA-STM and FA-STM models can only predict the pre-peak behavior of RC members, but the SMM model is capable of predicting both the pre-peak and post-peak behavior by taking into account the Hsu/Zhu ratios (Modified Poisson’s ratios of cracked RC members). The present study accomplishes two main tasks: (1) it formulates the SMM model in the context of fiber-based beam–column finite elements, and (2) it validates the finite element model by comparing its predictions with the experimental results of RC members.

To formulate the SMM model, three coordinate systems are typically assumed as shown in Fig. 2. The first coordinate system $(x, y)$ defines the local coordinate of the fiber element; the second coordinate system $(r, d)$ represents the applied principal stresses; while the third coordinate system $(\alpha, \beta)$ represents the concrete principal coordinate system in which the concrete shear stress $\tau_{12} = 0$. In the figure, $\alpha_1$ is the angle between the $x$-axis and the $\alpha$-axis, and $\alpha_2$ is the angle between the $x$-axis and the $\beta$-axis.

To rotate the stress and strain vectors from one system of axes to another, the following rotation matrix $R(\theta)$ is used:

$$R(\theta) = \begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta & \cos \theta \sin \theta & 2 \cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & -2 \cos \theta \sin \theta & \sin \theta \cos \theta \end{bmatrix}$$

(15)

whereas the angle $\theta$ is the angle between the two coordinate systems.

Three basic compatibility equations are defined as follows:

$$\varepsilon_x = a + b \cos 2\alpha_1$$

(16)

$$\varepsilon_y = a - b \cos 2\alpha_x$$

(17)

$$\gamma_{xy} = 2b \sin 2\alpha_x$$

(18)

where $a = 0.5(\varepsilon_t + \varepsilon_d)$ and $b = 0.5(\varepsilon_t - \varepsilon_d)$.

The principal strains in the 1–2 coordinate system are derived from the $x$–$y$ coordinate system as follows:

$$\varepsilon_1 = a + b \cos(2\alpha_1 - 2\alpha_t)$$

(19)

$$\varepsilon_2 = a - b \cos(2\alpha_1 - 2\alpha_t)$$

(20)

$$\gamma_{12} = 2b \sin(2\alpha_1 - 2\alpha_t)$$

(21)

The previous equations also lead to the following relations:

$$a = 0.25(\varepsilon_t + \varepsilon_y + \varepsilon_1 + \varepsilon_2)$$

(22)

$$\cos 2\alpha_1 = \frac{(\varepsilon_y - \varepsilon_x)}{2b} + \frac{\gamma_{xy}}{b} \cot 2\alpha_t.$$  

(23)

After substituting the value of the $\alpha_t$ and $b$ in Eq. (23), it becomes:

$$\cos 2\alpha_1 = \frac{(\varepsilon_t - \varepsilon_1)}{\varepsilon_t - \varepsilon_d}.$$  

(24)

Similar to the strain transformations (Eqs. (16)–(21)), stress transformation equations will be also later developed (Eqs. (29)–(46)). The transformation equations are graphically represented by Mohr’s stress and strain circles in Fig. 3.
4.1. Strains in circular sections

A circular cross section is more efficient to confine the concrete core than a rectangular section because the tension in curvilinear stirrups contributes to the confinement of the cross section. Circular cross sections are divided into a number of sectors along the circumferential direction (Fig. 4). The transverse reinforcement in each sector should follow the equilibrium and compatibility with the surrounding concrete. It is difficult to find the transverse reinforcement strain, stress and cross sectional area along the x-y coordinate system then later transformed to the x'-y' coordinate system for each section, the x'-y' coordinate system is derived by choosing the angle such that the z' axis is perpendicular to the transverse reinforcement alignment. The strain value in the x-y coordinate system is transformed to the x'-y' coordinate system with the help of the transformation matrix $A$ as follows:

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta' & -\sin \theta' \\ 0 & \sin \theta' & \cos \theta' \end{bmatrix}. \quad (25)$$

The strain in the x-y coordinate system is

$$[\varepsilon] = [A][s][A]^T. \quad (26)$$

The strain in the x'-y' coordinate system [s'] can be derived from Eqs. (25) and (26) as follows:

$$[s'] = [A][s][A]^T. \quad (27)$$

From Eq. (27), the strain perpendicular to the transverse reinforcement cross section $\varepsilon_{r'}$ is calculated as follows:

$$\varepsilon_{r'} = \cos^2(\theta'). \quad (28)$$

Eq. (27) is used to calculate the uniaxial transverse reinforcement stress in the x'-y' coordinate direction and this stress and the transverse reinforcement area at an angle $\theta'$ is transformed to the global x-y Cartesian coordinate system with the help of the transformation matrix $[A]$.

4.2. Evaluation of lateral strain

The equilibrium equations needed to evaluate the stresses in the x-y coordinate system $\{\sigma_x, \sigma_y, \tau_{xy}\}$ as a function of the principal stresses resisted by concrete $\{\sigma_1^c, \sigma_2^c, \tau_{12}^c\}$ and the reinforcing bar stresses $f_{x}$ and $f_{y}$ along the x and y directions respectively are:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \cos^2 \alpha_1 & \sin \alpha_1 \sin \alpha_1 & -2 \cos \alpha_1 \sin \alpha_1 \\ \sin^2 \alpha_1 & \cos^2 \theta & 2 \cos \alpha_1 \sin \alpha_1 \\ \cos \alpha_1 \sin \alpha_1 & -\cos \alpha_1 \sin \alpha_1 \cos^2 \alpha_1 - \sin^2 \alpha_1 \end{bmatrix} \times \begin{bmatrix} \sigma_1^c \\ \sigma_2^c \\ \tau_{12}^c \end{bmatrix} + \begin{bmatrix} \rho_{cxfx} \\ \rho_{cysy} \end{bmatrix}. \quad (29)$$

where $\rho_{cx}$ and $\rho_{cy}$ are the smeared steel ratio in the direction of x and y respectively.

The stresses in the x-y coordinate system can be written as a combination of concrete and steel stresses as follows:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \sigma_1^c + \sigma_{sx} \\ \sigma_2^c + \sigma_{sy} \\ \tau_{12}^c + \tau_{xy} \end{bmatrix}. \quad (30)$$

where superscript c stands for concrete and s stands for steel.

The second of the equilibrium equations in (29) is used to evaluate the lateral strain $\varepsilon_{y}$ in fiber $i$ owing to the fact that the value of $\sigma_y$ equals zero:

$$\sigma_1^c \sin^2 \alpha_1 + \sigma_2^c \cos^2 \alpha_1 + \tau_{12}^c \sin \alpha_1 \cos \alpha_1 + \rho_{fy} f_{yi} = 0 \quad (31)$$

which can also be written as:

$$\sigma_1^c A_{cy} + \sigma_2^c A_{sy} = 0 \quad (32)$$

whereas $\sigma_{yi}$ is the concrete stress in the transverse direction of fiber $i$ and is equal to the sum of the first three terms in Eq. (31), $\sigma_{yi}$ is the steel stress in the transverse direction of fiber $i$, $A_{cy}$ and $A_{sy}$ are respectively the area of concrete and steel within the spacing.
5 of the transverse reinforcement as shown in Fig. 5, $\rho_{ty}^i$ is the ratio of steel to concrete area in the transverse direction of fiber $i$, and $f_{ty}^i$ is the transverse steel bar stress which equals $\sigma_{ty}^i$.

Due to the nonlinear behavior of the concrete and steel, an iterative procedure is needed to determine the lateral strain $\varepsilon_{y}$ that satisfies Eq. (32). An initial value for $\varepsilon_{y}$ is assumed at each fiber, and the iterations proceed until Eq. (32) is satisfied. The process for evaluating the concrete stress from the fiber strain through the concrete constitutive model is described next.

### 4.3. Evaluation of concrete stress

The typical concrete stress–strain curves are derived from uniaxial tests, and hence the biaxial strains in the $x$-$y$ direction $[\varepsilon_x, \varepsilon_y, \gamma_{xy}]^T$ need to be converted to equivalent uniaxial strains in the $1$–$2$ direction $[\bar{\varepsilon}_1, \bar{\varepsilon}_2, \gamma_{12}]^T$ so as to calculate the concrete stresses.

Initially, the direction of cracks is calculated based on a rotating angle $\alpha_{1}$, derived from the strain state. At a rotating principal angle $\alpha_{1}$, the concrete shear stress $\tau_{12}^c = 0$ (Fig. 3) and the value of $\alpha_{1}$ is:

$$\tan 2\alpha_{1} = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}. \tag{33}$$

After evaluating the $\varepsilon_{y}$ term that satisfies the equilibrium conditions, the calculated principal angle $\alpha_{1}$ from the known stress state is evaluated as follows:

$$\tan 2\alpha_{1} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}. \tag{34}$$

The biaxial principal strains are then evaluated as follows:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ 0.5\gamma_{12} \end{bmatrix} = [R(\alpha_{1})] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ 0.5\gamma_{xy} \end{bmatrix}. \tag{35}$$

### 5. Uniaxial constitutive relationships of the materials

#### 5.1. Concrete model

The concrete model describes the cyclic uniaxial constitutive relationships of cracked concrete in compression and tension. The envelop curve of the concrete in compression follows the modified Kent and Park model [35] which offers a good balance between simplicity and accuracy. The model implemented in this study has the following characteristics: First, the effect of softening in both the stress and strain curves was considered. Second, the successive degradation of stiffness of both the unloading and reloading curves

$$\begin{cases} \varepsilon_{x} = 0.2 + 850\varepsilon_{yf} & \varepsilon_{yf} \leq \varepsilon_{yd} \\ \mu_{12} = 1.9 & \varepsilon_{yf} > \varepsilon_{yd} \end{cases} \tag{36}$$

where $\varepsilon_{yd}$ is defined as the strain in the reinforcement that yields first and $\varepsilon_{yd}$ is the yield strain of the reinforcing steel.

After cracking, Hsu/Zhu ratio $\mu_{12}$ lies outside the typical range of zero to 0.5 for Poisson ratios of continuous materials. Before cracking the Hsu/Zhu ratio $\mu_{21} = 0.2$ and, after cracking the Hsu/Zhu ratio $\mu_{21} = 0$, indicating that the tensile strain does not have any effect on the compressive strain.

The equivalent uniaxial strains are derived from the biaxial principal strains with Hsu/Zhu ratios $(\mu_{12}, \mu_{21})$ as follows:

$$\begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ 0.5\gamma_{12} \end{bmatrix} = [\mu] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ 0.5\gamma_{xy} \end{bmatrix}. \tag{37}$$

where

$$[\mu] = \begin{bmatrix} 1 & \mu_{12} & 0 \\ 1 - \mu_{12}\mu_{21} & 1 - \mu_{12}\mu_{21} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

The equivalent uniaxial principal strain in the longitudinal reinforcement is given by:

$$\varepsilon_{ax} = \left(1 - \frac{\mu_{12}\mu_{21}}{1 - \mu_{12}\mu_{21}}\right)\varepsilon_{x} + \frac{\mu_{12}}{1 - \mu_{12}\mu_{21}}\varepsilon_{y} \cos^2(\alpha_{1}) + \frac{1}{1 - \mu_{12}\mu_{21}}(\varepsilon_{1} + \varepsilon_{2}) \sin^2(\alpha_{1}) - \gamma_{12} \sin(\alpha_{1}) \cos(\alpha_{1}). \tag{39}$$

The equivalent uniaxial principal strain in the transverse reinforcement is given by:

$$\varepsilon_{ay} = \left(1 - \frac{\mu_{12}\mu_{21}}{1 - \mu_{12}\mu_{21}}\right)\varepsilon_{y} + \frac{\mu_{12}}{1 - \mu_{12}\mu_{21}}\varepsilon_{x} \sin^2(\alpha_{1}) + \frac{1}{1 - \mu_{12}\mu_{21}}(\varepsilon_{1} + \varepsilon_{2}) \cos^2(\alpha_{1}) + \gamma_{12} \sin(\alpha_{1}) \cos(\alpha_{1}). \tag{40}$$

The equivalent uniaxial longitudinal steel stress $f_{sx}$ and transverse steel stresses $f_{sy}$ are calculated from the steel strains $\varepsilon_{sx}$ and $\varepsilon_{sy}$ respectively through a Menegotto–Pinto stress–strain relationship [34].

The current equivalent uniaxial strains $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ are individually used to calculate the concrete stresses $\sigma_{x}^{c}$ and $\sigma_{y}^{c}$ respectively in the principal direction from the stress–strain relation of the uniaxial concrete material. Equivalent uniaxial steel strains $\bar{\varepsilon}_{ax}$ and $\bar{\varepsilon}_{ay}$ are used to calculate the steel stresses $f_{sa}$ and $f_{sy}$ respectively from the stress–strain relation of the Menegotto–Pinto relation as described in the next section.

Fig. 5. Concrete strut and transverse reinforcement equilibrium.
effect under cyclic loads and agrees very well with experimental results.

6. Concrete biaxial constitutive relations

The concrete constitutive models in principal directions 1–2 is evaluated based on the following three strain conditions:

6.1. 1-tension, 2-compression

In this case, the equivalent uniaxial strain of concrete \( \varepsilon_1 \) in principal direction 1 is in tension, and the equivalent uniaxial strain \( \varepsilon_2 \) in principal direction 2 is in compression. Due to this condition, the uniaxial concrete stress \( \sigma_1 \) in direction 1 is calculated from \( \varepsilon_1 \), and is not a function of the perpendicular concrete strain \( \varepsilon_2 \). The compressive strength in principal direction 2 however, \( \sigma_2 \) will soften due to the tension in the orthogonal direction. Zhu and Hsu [11] derived a softening equation in the tension–compression region, which is implemented in the current model \( \text{Fig. 9} \), and is based on panel testing as proposed in \[1\]. The equation for compressive strength reduction factor proposed in \[11\] is:

\[
\zeta = \left( \frac{5.8}{\sqrt{f'_c (\text{MPa})}} \right) \leq 0.9 \left( \frac{1}{\sqrt{1 + 400\varepsilon_1}} \right) \left( 1 - \left| \alpha_r \right| \right) \cdot 24^\circ.
\]

From \text{Fig. 6},

\[
\alpha_r = 0.5 \tan^{-1} \left( \frac{\gamma_{12}}{\varepsilon_1 - \varepsilon_2} \right).
\]

The softening coefficient \( \zeta \) value is limited to 0.9, because the uniaxial concrete compressive strength \( f'_c \) is calculated from standard cylinder tests, while from the panel experiments at the University of Houston it was observed that the concrete strength does not reach \( f'_c \). The reason is due to size effect, loading rate effect, and shape factor which have ample effect on the concrete compressive strength \( f'_c \). The ultimate stress in the orthogonal directions is therefore \( f''_c \) at a softened strain \( \zeta \varepsilon_0 \) (Fig. 9), where \( \zeta \) is the softening coefficient, \( \alpha_r \) is the deviation angle which is the difference between the applied stress angle \( \alpha_1 \) and the rotating angle \( \alpha_r \). \( \varepsilon_1 \) is the lateral tensile strain, \( \varepsilon_0 \) is the concrete strain at peak compressive strength \( f''_c \), and \( \zeta f''_c \) is the softened concrete compressive strength.

6.2. 1-tension, 2-tension

The equivalent uniaxial strain of concrete \( \varepsilon_1 \) in direction 1 is in tension, and the equivalent uniaxial strain \( \varepsilon_2 \) of concrete in direction 2 is also in tension. In this case, the uniaxial concrete
The ultimate stresses in the orthogonal directions are:

\[ \sigma^{1}_{\text{f}} \] and \[ \sigma^{2}_{\text{f}} \] are functions of the orthogonal concrete strains \( \epsilon^{1}_{1} \) and \( \epsilon^{2}_{2} \) respectively.

6.3. 1-compression, 2-compression

The equivalent uniaxial strains of concrete in principal directions 1 and 2 are both in compression. The current research uses Vecchio's [38] simplified version of the Kupfer et al. biaxial compression strength equation [39]. The concrete compressive strength increase in one direction depends on the confining stress in the orthogonal direction (Fig. 10). The strength enhancement and increase in ductility are dependent on the biaxial compressive stresses. Concrete in compression exhibits lateral expansion and increase in the value of Poisson ratio. An upper limit of 0.5 has been considered for Poisson ratio.

The principal stresses in the two orthogonal directions are denoted by \( \sigma^{1}_{\text{f}} \) and \( \sigma^{2}_{\text{f}} \) and their corresponding strains are referred by \( \epsilon^{1}_{1} \) and \( \epsilon^{2}_{2} \). The equations for the biaxial compression failure surface are:

\[
K_{1} = 1 + 0.92 \left( -\frac{\sigma^{1}_{\text{f}}}{f_{c}^{1}} \right) - 0.76 \left( -\frac{\sigma^{2}_{\text{f}}}{f_{c}^{2}} \right)^{2} \tag{43}
\]

\[
K_{2} = 1 + 0.92 \left( -\frac{\sigma^{2}_{\text{f}}}{f_{c}^{2}} \right) - 0.76 \left( -\frac{\sigma^{1}_{\text{f}}}{f_{c}^{1}} \right)^{2} \tag{43}
\]

where \( K_{1} \) and \( K_{2} \) are the biaxial strength magnification factors. The ultimate stresses in the orthogonal directions are:

\[
\sigma^{1}_{\text{f}} = K_{1}\sigma^{1}_{\text{f}} \tag{44}
\]

\[
\sigma^{2}_{\text{f}} = K_{2}\sigma^{2}_{\text{f}} \tag{45}
\]

The ultimate strains in the orthogonal directions are:

\[
\epsilon^{1}_{1} = K\epsilon^{1}_{1} \tag{46}
\]

\[
\epsilon^{2}_{2} = K\epsilon^{2}_{2} \tag{47}
\]

where \( K \) is the uniaxial compressive strain of the concrete at the peak stress.

After evaluating the concrete stresses from the biaxial constitutive laws, the fiber stiffness matrix is constructed, as described in the next section.

7. Fiber state determination

The concrete model is simplified with a plane stress orthotropic material. It has mutually perpendicular planes of elastic symmetry. Directions 1 and 2 are the local principal material axes that are normal to the planes of symmetry (Fig. 2).

With the equivalent uniaxial strains, the stiffness values \( \bar{E}^{1}_{1} \) and \( \bar{E}^{2}_{2} \) are determined from a material uniaxial stress–strain diagram. The material constitutive equation is:

\[
[\sigma^{1}_{12}] = [D^{1}_{\text{f}}][\epsilon^{1}_{12}] \tag{46}
\]

where \( [\sigma^{1}_{12}] \) is the local concrete stress vector, \( [\epsilon^{1}_{12}] \) is the local principal strain vector, and \( [D^{1}_{\text{f}}] \) is the local uniaxial concrete material secant stiffness matrix in the principal direction:

\[
[D^{1}_{\text{f}}] = \begin{bmatrix}
1 - \mu_{12}\mu_{21} & \frac{\bar{E}^{1}_{1}}{\mu_{21}} & 0 \\
\frac{\bar{E}^{1}_{1}}{\mu_{21}} & 1 - \mu_{12}\mu_{21} & 0 \\
0 & 0 & \bar{G}_{12}
\end{bmatrix}. \tag{47}
\]

\( \bar{G}_{12} \) is the concrete shear modulus which relates the shear stress \( \tau^{1}_{12} \) to the shear strain \( \epsilon^{1}_{12} \) in the 1–2 direction.

Based on a smeared crack concept from Mohr circles of stresses and strains (Fig. 3), and assuming the principal stress and principal strain directions of concrete to coincide with each other, \( \bar{G}_{12} \) is given as:

\[
\bar{G}_{12} = \frac{\tau^{1}_{12} - \epsilon^{1}_{12}}{\epsilon^{1}_{1} - \epsilon^{2}_{2}}. \tag{48}
\]

The concrete orthotropic stiffness matrix in the global x–y direction \([D^{1}_{\text{f}}] \) is evaluated through the rotation matrix \( R \):

\[
[D^{1}_{\text{f}}] = [R(-\alpha_{1})][D^{1}_{\text{f}}][R(\alpha_{1})]. \tag{49}
\]

The local uniaxial reinforcement material stiffness matrix in the direction of reinforcement is given by:

\[
[D^{1}_{\text{f}}] = \begin{bmatrix}
\rho_{\text{sx}}\bar{E}_{\text{sx}} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad \text{and} \quad [D^{1}_{\text{f}}]^y = \begin{bmatrix}
\rho_{\text{sy}}\bar{E}_{\text{sy}} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}. \tag{50}
\]

where \( [D^{1}_{\text{f}}] \) is the longitudinal steel local stiffness matrix and \( [D^{1}_{\text{f}}]^y \) is the transverse steel local stiffness matrix, \( \rho_{\text{sx}} \) is the smeared area of the longitudinal steel in fiber i, and \( \rho_{\text{sy}} \) is the smeared area of the transverse steel, and \( \bar{E}_{\text{sx}} \) and \( \bar{E}_{\text{sy}} \) are the uniaxial steel stiffnesses evaluated from the Menegotto–Pinto steel model along the longitudinal and transverse directions respectively.

The reinforcement stiffness matrix in the global direction, \([D^{1}_{\text{f}}] \) is the sum of the longitudinal steel global stiffness matrix \([D^{1}_{\text{f}}]^x\) and the transverse steel global stiffness matrix \([D^{1}_{\text{f}}]^y\) which are defined below:

\[
[D^{1}_{\text{f}}] = [R(0\deg)]D^{1}_{\text{f}}[R(-\alpha_{1})]\mu[R(\alpha_{1})]. \tag{51}
\]

\[
[D^{1}_{\text{f}}]^y = [R(-90\deg)]D^{1}_{\text{f}}[R(90\deg - \alpha_{1})]\mu[R(\alpha_{1})]. \tag{52}
\]

The stiffness matrix including concrete and transverse steel terms is evaluated from the concrete stiffness \([D^{1}_{\text{f}}]\) as well as the transverse steel stiffness \([D^{1}_{\text{f}}]^y\) as follows:

\[
[D^{1}_{\text{f}}] = \begin{bmatrix}
D^{1}_{11} & D^{1}_{12} & D^{1}_{13} \\
D^{1}_{21} & D^{1}_{22} & D^{1}_{23} \\
D^{1}_{31} & D^{1}_{32} & D^{1}_{33}
\end{bmatrix}. \tag{53}
\]

The total global stiffness matrix is non-symmetric since the off-diagonal values are affected by the Hsu/Zhu Poisson ratios which depend on the stress state.

Finally, a new process for determination of the sectional and elemental stiffness matrices derived from fiber discretization is proposed in the next section.
8. Sectional and element stiffness and force evaluation

The stress and strain in the global coordinate system are related as follows:

$$ \begin{pmatrix} \sigma_{cx} \\ \sigma_{cy} \\ \tau_{xy} \end{pmatrix} = \begin{bmatrix} D_{gl} \end{bmatrix}^T \begin{pmatrix} \varepsilon_{cx} \\ \varepsilon_{cy} \\ \gamma_{xy} \end{pmatrix} $$

where $\sigma_{cx}$ is the longitudinal stress in a concrete fiber, $\sigma_{cy}$ is the total transverse fiber stress due to concrete and steel, and $\tau_{xy}$ is the total fiber shear stress.

The proposed element solution is based on a plane section hypothesis. The sectional degree of freedom term corresponding to the transverse strain $\gamma_{xy}$ is condensed out by first rewriting (53b) as:

$$ \begin{bmatrix} k_{as,x} & k_{as,y} \\ k_{as,x,y} & k_{as,x} \end{bmatrix} \begin{pmatrix} \varepsilon_{as,x} \\ \varepsilon_{as,y} \end{pmatrix} = \begin{pmatrix} \sigma_{as,x} \\ \sigma_{as,y} \end{pmatrix} $$

where

$$ k_{as,x} = [D_{21} D_{22}] \quad k_{as,y} = \begin{bmatrix} D_{11} & D_{13} \\ D_{13} & D_{33} \end{bmatrix} \quad k_{as,x,y} = D_{12} $$

$$ \varepsilon_{as,x} = \varepsilon_{x} \quad \varepsilon_{as,y} = \varepsilon_{y} $$

The condensed fiber stiffness is:

$$ [k_{as}] = [k_{as}] - [k_{as,x,y}] \frac{k_{as,x,y}}{k_{as}} $$

The condensed fiber stresses are

$$ [\sigma_{fiber}] = \begin{bmatrix} \tilde{\sigma}_{cx} \\ \tilde{\sigma}_{cy} \end{bmatrix} = \begin{pmatrix} \bar{\sigma}_{cx} \\ \bar{\sigma}_{cy} \end{pmatrix} - \begin{bmatrix} D_{12} \\ D_{32} \end{bmatrix} \begin{pmatrix} \sigma_{cx} \\ \sigma_{cy} \end{pmatrix} $$

The fiber strains are derived from the section strains as follows:

$$ \begin{pmatrix} \varepsilon_{x} \\ 0.5 \gamma_{xy} \end{pmatrix} = \begin{bmatrix} 1 & 0 & -y \\ 0.5 & 0 & 0 \end{bmatrix} \begin{pmatrix} \varepsilon_{0} \\ \gamma_{xy} \end{pmatrix} $$

The transformation matrix to transform the fiber stiffness to the section stiffness is therefore:

$$ [T] = \begin{bmatrix} 1 & 0 & -y \\ 0 & 0.5 & 0 \end{bmatrix} $$

The contribution of concrete to the section stiffness is:

$$ [(K_{section})_{x}] = \sum_i ([T]^T [k_{as}] [T]) A_{ax} $$

which results in:

$$ [K_{section}] = \sum_i \begin{bmatrix} D_{11} - D_{12} D_{21} & 0.5 \left(D_{11} - D_{12} D_{21}\right) & -y \left(D_{11} - D_{12} D_{21}\right) \\ 0.5 \left(D_{11} - D_{12} D_{21}\right) & D_{13} - D_{12} D_{22} & -0.5 y \left(D_{13} - D_{12} D_{22}\right) \\ -y \left(D_{11} - D_{12} D_{21}\right) & -0.5 y \left(D_{13} - D_{12} D_{22}\right) & y^2 \left(D_{11} - D_{12} D_{21}\right) \end{bmatrix} A_{ax} $$

The sectional stiffness due to the longitudinal reinforcement is given by:

$$ [(K_{section})_{x}] = \sum_i ([T]^T [D_{gl}] [T]) A_{ax} $$

The sectional forces due to the concrete fiber are given as:

$$ ([F_{section})_{c}] = \sum_i ([T]^T [\sigma_{fiber}] A_{cx} $$

The sectional forces due to the longitudinal steel fiber are given by:

$$ ([F_{section})_{x}] = \sum_i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & y \end{bmatrix} \begin{pmatrix} \sigma_{cx} \\ \sigma_{cy} \end{pmatrix} A_{ax} $$

The section forces in the longitudinal, transverse, and rotational directions are given respectively by:

$$ [F_{x}] = \sum_i \bar{\sigma}_{cx} A_{cx} + \sum_i \sigma_{cx} A_{cx} $$

$$ [F_{y}] = \sum_i \bar{\sigma}_{cy} A_{cy} + \sum_i \sigma_{cy} A_{cy} $$

$$ [M_{xy}] = \sum_i -y \bar{\sigma}_{cx} A_{cx} + \sum_i -y \sigma_{cx} A_{cx} $$

The total stiffness of the section is derived from the sum of concrete and steel stiffnesses:

$$ [K_{section}] = \sum_i (K_{section})_{c} + \sum_i (K_{section})_{x} $$

where $nc$ and $ns$ are the number of concrete and longitudinal steel fibers in a section.

The total force of the section is the sum of concrete and steel forces in their respective directions:

$$ [F_{section}] = \sum_i (F_{section})_{c} + \sum_i (F_{section})_{x} $$

The element stiffness and forces are calculated with numerical integration respectively as follows:

$$ [K_{element}] = \int \sum_{i=1}^{n_{ip}} B(x)^T (x) [K_{section}] (x) B(x) \, dx $$

$$ [F_{element}] = \int \sum_{i=1}^{n_{ip}} W_i B(x_i)^T (x) [F_{section}] (x) B(x_i) \, dx_i $$

where, $W$ is the Gaussian weight, $n_{ip}$ is the number of integration points, and $j$ is the Jacobian.

9. Nonlinear analysis procedure

A new iterative tangent stiffness procedure is proposed to perform the nonlinear analysis of shear-critical reinforced concrete structures. A flow chart for the iterative solution under load increments using the Newton–Raphson method is described in Fig. 11. Throughout the procedure, the tangent material constitutive matrix $[D_{gl}]^T$ is determined first, and the tangent element stiffness matrix $[K_{element}]$ and the element resisting force increment vector $[P_{element}]$ are calculated. Then, the global stiffness matrix $[K]$ and global resisting force increment vector $[R_{g}]$ are...
assembled. In each iteration, the material constitutive matrix \([D_{el}]\), the element tangent stiffness matrix \([K_{element}]\), and the global stiffness matrix \([K]\) are iteratively refined until convergence is achieved. It is noted that the two additional iterative loops are defined to obtain the section stiffness matrix \([K_{section}]\) because the principal stress angle \(\alpha_1\) and the transverse steel strain \(\varepsilon_t\) are unknown values before determining the section stiffness matrix \([K_{section}]\).

10. Numerical correlations with experimental results

10.1. Xiao and Martirosyan shear-critical column

The proposed element was validated by modeling a high strength reinforced concrete squat column (Column HC4-BL 16-T6-0.2P) tested by Xiao and Martirosyan [40] at the University of Southern California (Fig. 12(a)). An axial load of 1068 kN (240 kip) was applied constantly to the column. Rotations were fixed at the column bottom and top so that the column deforms anti-symmetrically with respect to the mid-height under combined axial and lateral loading (Fig. 12(b)). The width of the column section is 254 mm (10 in.) and its depth is also 254 mm (10 in.). The longitudinal reinforcement consists of 8 No. 16 (15.9 mm (0.63 in.) diameter) bars, uniformly spaced along the perimeter with a clear cover of 13 mm (0.51 in.). No. 6 (6.4 mm (0.25 in.) diameter) steel stirrups are provided with a spacing of 51 mm (2 in.).

An average concrete compressive strength \(f'_c = 86 \text{ MPa (12.5 ksi)}\) is used to analyze the reinforced concrete column. Yield stresses of \(f_y = 510 \text{ MPa and } 449 \text{ MPa (74 and 65.12 ksi)}\) are being used for the longitudinal and transverse reinforcements respectively. The Young’s moduli of concrete and steel are \(E_c = 45,500 \text{ MPa (6600 ksi)}\) and \(E_s = 200,000 \text{ MPa (29,000 ksi)}\)

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**Fig. 11.** Nonlinear analysis procedure.
respectively. A tensile strength of the concrete of 3 MPa is used in the analysis, and the tension stiffening of concrete and smeared stress–strain relationship of reinforcement are being implemented in the model according to Belarbi and Hsu [2].

The column section is subdivided into 12 fibers and modeled with only one element along the length. A Gauss–Labatto integration scheme with five integration points is used in the analysis. These numbers of sections and fibers proved to be sufficient to reach a converged solution. The column is analyzed by assuming that the transverse steel is smeared along the length of the column because the spacing of the stirrups is only 51 mm (2 in.). Longitudinal bars are not being smeared because of the discrete nature of the arrangement of the reinforcement.

Column HC4-8L 16-T6-0.2P has low shear reinforcement ($\rho_s = 1.63\%$) and is tested under cyclic uni-axial bending. The experimental and analytical cyclic force–displacement results are shown in Figs. 13 and 14. Fig. 13 refers to the Bernoulli type flexural element with infinite shear strength which tends to overestimate the strength and ductility, while Fig. 14 refers to the shear element which shows a very good correlation with the experimental results. The shear model showed a flexural-shear failure mode, where shear failure was observed at the column ends with the increase of transverse strains, followed by core degradation and stirrups yielding. The new element predicted accurately the amount of energy dissipation and displaced shape. Because of the formation of a plastic hinge after yielding of the reinforcement, the resisting load is stable with a capacity of 300 kN (68 kip) until a ductility level of 4.

### 10.2. Missouri ST short column

The model was used to simulate the behavior of a short reinforced concrete bridge column (Column H/D(3)-T/M(0)) tested by Prakash et al. [41] at the Missouri University of Science and Technology. The column has a length-to-diameter ratio of 3. The actual test specimen has a 610 mm diameter and is 1.83 m long from the top of the footing to the centerline of the applied load. A 70 mm hole in the center of the column cross section was used to run seven high-strength steel strands that were stressed to apply the axial load. The load was applied at the top of the column using two hydraulic actuators in a displacement-controlled mode. The reinforcement consisted of 12 No. 8 (25.4 mm) longitudinal bars, and No. 4 (12.7 mm) spiral transverse reinforcement spaced at 70 mm.

A concrete compressive strength of $f'_c = 25.8$ MPa and tensile strength of $f_t = 3.5$ MPa are used to analyze the reinforced concrete column. A Yield stress of longitudinal and transverse reinforcement of $f_y = 458$ MPa and $f_y = 450$ MPa respectively is used in the analysis. The Young’s modulus of concrete and steel are $E_c = 27,600$ MPa and $E_s = 200,000$ MPa, respectively.

Column H/D(3)-T/M(0) is the control specimen, and is tested under cyclic uniaxial bending. The force-based shear element model was used to simulate the experimental monotonic force–displacement response with only one element, as shown in Fig. 15. The figure reveals the analytical model can capture the global response rather very well. The column capacity of 490 kN was also accurately estimated by the model. The model failed in a flexure-shear mode followed by core degradation. The longitudinal reinforcement yielded at 360 kN, but the transverse reinforcement did not yield because the higher ratio of the spiral reinforcement increased the column confinement. Due to the low length-to-diameter ratio, shear still affected the behavior, and the observed crack pattern was not completely horizontal. A good correlation for the initial stiffness, yielding of the reinforcement, ultimate load capacity, and ductility is clearly observed between the analytical and experimental results.
Fig. 15. Monotonic shear force–displacement of Missouri S&T short column.

The monotonic envelope of the response was also evaluated using the commercial FE program DIANA (Displacement ANALyzer) as reported by Prakash et al. [41]. Fig. 15 reveals that the force-based shear model was able to capture the behavior with only one finite element, while the DIANA model with 3200 solid brick elements (HX24L) was unable to accurately capture the resisting force–displacement response. The proposed shear element is also more efficient because the column was analyzed in about two minutes while the DIANA model took more than thirty minutes for completing the analysis.

10.3. UNR shear-critical column

The model was used to simulate the dynamic behavior of the circular shear-critical RC bridge column 9S1 tested by Laplace et al. [42] at the University of Nevada, Reno. The column was subjected to the real time 1940 Imperial Valley (El Centro) earthquake excitation. An axial load of 356 kN (80 kip) was applied constantly to the column. The column is loaded in double curvature. The actual test specimen has a 406.44 mm (16 in.) diameter and is 1.22 m (48 in.) long from the top of the footing to the centerline of the applied load. The reinforcement consisted of sixteen No. 6 longitudinal bars, and 6.5 mm (0.25 in.) spiral transverse reinforcement spaced at 38.1 mm (1.5 in.) as shown in Fig. 16(a) and (b).

A concrete compressive strength of $f'_c = 36.7$ MPa (5.3 ksi) and tensile strength of $f_t = 2.0$ MPa (0.3 ksi) are used to analyze the reinforced concrete column. Yield stresses of longitudinal and transverse reinforcement of $f_y = 458.2$ MPa (65.0 ksi) and $f_y = 397.4$ MPa (57.6 ksi) respectively are used in the analysis. The Young’s moduli of concrete and steel are $E_c = 36,500$ MPa (5300 ksi) and $E_s = 200,000$ MPa (29,000 ksi) respectively. A tensile strength of the concrete of 2 MPa was used in the analysis and the tension stiffening of the concrete was assumed according to Belbari and Hsu [2].

The 9S1 column is reinforced with 3.5% longitudinal steel and only with 0.92% transverse reinforcement. The double curvature boundary condition and the higher longitudinal reinforcement have augmented the shear effect while reducing the flexural demand.

The circular transverse reinforcement under tension leads to the additional shear enhancement of the hoops. The proposed angular section discretization allowed the model to simulate this phenomenon in the circular cross section. The 9S1 Column was analyzed with the $1 \times$ El Centro (Fig. 16(c)) at PGA of 0.32 g and $2.5 \times$ El Centro at PGA of 0.81 g earthquake motions. The comparison of the experimental results with the flexural element and the currently developed shear element are presented in Figs. 17–26. A Rayleigh damping with a 5% damping ratio for the first mode was considered in the analysis. A time step of 0.01 s has been used in the earthquake analysis.

At $1 \times$ El Centro motion, the flexural element was unable to predict the displacement amplitude accurately (Fig. 17) but the shear element was fairly accurate in predicting them (Fig. 18). The flexural element was also very stiff, less ductile and unable to predict the resisting load (Fig. 19), but the shear element was able to predict the experimental displacement and the resisting load accurately (Fig. 20). At $1.0 \times$ El Centro earthquake motion the longitudinal steel bar did not yield (Fig. 25) and the strain of
the transverse steel (Fig. 26) was much smaller than that of the longitudinal steel.

At 2.5 \times \text{El Centro} earthquake motion the flexural element was also unable to accurately predict the displacement amplitude and the resisting load capacity (Figs. 21 and 23), while the shear element was able to predict well the response (Figs. 22 and 24). At this El Centro motion, the longitudinal bar at the gauge 1 location shown in Fig. 25 yielded, which was fairly accurately simulated using the numerical model. The transverse steel bar at the gauge 2 location shown in Fig. 26 also yielded. The shear model accurately predicted the transverse strains along the plastic hinge length, but slightly overestimated the values near the column ends. This is in part due to the presence of the loading blocks which were not accounted for in the finite element model.

11. Conclusion

The paper presents a new element for dynamic analysis of shear-critical reinforced concrete structures. The element is based on the force method of analysis with the section discretization into fibers, and accounts for shear effects through the use of the Softened Membrane Model. The proposed model is able to capture the interaction between axial, flexural, and shear responses. The implementation of the element in a general purpose finite element
program was presented and the numerical iterative algorithm was discussed in detail. Correlation studies with specimens tested under quasi-static and shake table excitations were conducted. The results showed that the model is able to reasonably capture the response of shear-critical RC elements and predict the proper failure mode. A good correlation for the initial stiffness, yielding of the reinforcement, ultimate load capacity, and ductility was typically observed. Local behavior such as longitudinal and transverse steel strains were also reasonably predicted. In addition, the proposed element proved to be much more accurate and computationally efficient than commercial finite element programs using solid brick elements.

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