Nonlinear finite element modeling of beams on two-parameter foundations

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1. Introduction

The inelastic response of shallow and raft foundations is significantly complex due to the behavior of the surrounding semi-infinite soil media. Winkler’s model [1] is the simplest element that accounts for the behavior of both the foundation and soil. The Winkler approach models the soil as a single layer, and assumes that the foundation reaction at a particular point is proportional to the soil displacement. The Winkler model is considered therefore a single-parameter model with the spring’s elasticity as its only parameter. While this model is associated with closely spaced independent elastic springs, in reality these springs should be dependant on each other. To address these drawbacks, several modified approaches have been proposed such as the ones developed by Filonenko-Borodich [2], Hetenyi [3], Pasternak [4], and Vlasov and Leontiev [5]. These models belong to the family of multiple-parameter foundation models because, in addition to the first parameter spring’s elasticity, they assumed a second parameter that accounts for the effect of the adjacent soil medium. In this paper, the Pasternak [4], and Vlasov and Leontiev [5] models were adopted. In the Pasternak model, the shear interaction between the Winkler spring elements is considered. The shear layers consist of incompressible vertical elements which deform only in transverse shear. In the Vlasov and Leontiev model, the second parameter is considered by extending the Pasternak spring elements with a consideration of the effect of the soil on both sides of the beam. The stiffness matrix of an elastic beam on multiple-parameter foundation element can be derived based on different orders of displacement shape functions or by using the exact displacement function obtained from the solution of the differential equations governing the behavior. Biot [6] studied the foundation as an elastic continuum and derived an exact solution to an infinite beam under a concentrated load. Kerr [7] studied the foundation response using an elastic continuum approach by connecting each two spring layers with an interconnecting shear layer. Reissner [8] formulated the problem of an elastic plate on an elastic foundation by assuming a transition condition at the interior of the foundation layer along the cylindrical surface. Harr et al. [9] analyzed beams on elastic foundations based on Vlasov general variational method in which the elastic foundation is represented by a single layer. Yang [10] introduced a numerical iterative procedure on the basis of the finite element method for analyzing plates on elastic foundations. Zhaohua and Cook [11] developed the finite element formulation of an elastic beam on two-parameter foundation using both, an exact displacement function, and a cubic displacement function for the case of distributed loads acting along the entire beam length. Chiwanga and Valsangkar [12] extended the approach for the case of a generalized distributed load. Sharma and Giger [13] developed the stiffness matrix and nodal-action column vectors for a Timoshenko beam on two-parameter foundation element. Razaqpur and Shah [14] derived the stiffness matrix and nodal load vector of an element representing a beam on two-parameter elastic foundation using polynomial displacement shape functions. Vallabhan and Das [15] developed a unique iterative technique to determine the values of the Vlasov parameters used in [9]. De Rosa [16] studied the free vibration of Timoshenko...
The equilibrium of an element of length $dx$ of a beam element resting on a two-parameter foundation, as shown in Fig. 1, is given by:

$$V_x + (w - t_f) = 0$$  \hspace{1cm} (1)

$$M_x + V = 0$$  \hspace{1cm} (2)

where $V$ and $M$ denote the shear force and bending moment, respectively, $t_f$ is the foundation force per unit length, $w$ denotes the distributed load on the beam, and a comma denotes a derivative. According to the two-parameter foundation hypothesis, and assuming linear soil behavior, the foundation force per unit length is related to the transverse displacement as follow:

$$t_f = k_f v(x) - k_m v_{,xx}(x)$$  \hspace{1cm} (3)

where $k_f$ is the Winkler’s modulus and $k_m$ is a second parameter that depends on both the soil and foundation characteristics. For inelastic behavior, both parameters will be based on nonlinear functions as will be described later. The foundation force term corresponding to the second parameter can be viewed as an additional moment resistance provided by the foundation following elementary thin-plate theories. Accordingly, the foundation moment per unit length, assuming linear behavior, is defined as follows:

$$t_m = k_m v_x$$  \hspace{1cm} (4)

From Eqs. (1)–(3):

$$M_{xx} - t_{mx} + t_f - w = 0$$  \hspace{1cm} (5)

The values of the two foundation parameters $k_f$ and $k_m$ are typically evaluated based on two-parameter equations [5]. For an approximate analysis, Vlasov and Leonov [5] assumed the transverse displacement $v(x, y)$ as a function of a vertical surface displacement $v(x)$ and a shape function $h(y)$. These equations were derived for a beam of finite width resting on an elastic foundation layer in a plane strain condition, as shown in Fig. 2:

$$v(x, y) = v(x)h(y)$$  \hspace{1cm} (6)

where $h(0) = 1, h(H) = 0; H$ being the depth of the soil layer, and

$$h(y) = \frac{\sinh \gamma (1 - \frac{y}{H})}{\sinh \gamma}$$  \hspace{1cm} (7)

Based on these assumptions, the parameters $k_f$ and $k_m$ are evaluated as:

$$k_f = \frac{(1 - \nu_2)E_b b}{(1 + \nu_1)(1 - 2\nu_2)} \frac{\gamma \sinh \gamma \cosh \gamma + \gamma^2}{2 \sinh^2 \gamma}$$  \hspace{1cm} (8)

$$k_m = \frac{E_b b H}{2(1 + \nu_1)} \frac{\sinh \gamma \cosh \gamma - \gamma}{2 \sinh^2 \gamma}$$  \hspace{1cm} (9)

The $\gamma$ parameter is a coefficient that determines the rate of decrease of the displacements over the depth of the foundation and is evaluated from the equation given below [5]:

$$\left(\frac{\gamma}{H}\right)^2 = \frac{(1 - 2\nu_2)}{2(1 - \nu_1)} \int_0^H v(x) \text{d}x + 0.5 \frac{\sqrt{\nu_1}}{\sqrt{\nu_2}} \int_0^L t^2(0) + t^2(L)$$

$$2(1 - \nu_1) \int_0^H v^2(x) \text{d}x + 0.5 \frac{\sqrt{\nu_1}}{\sqrt{\nu_2}} \int_0^L t^2(0) + t^2(L)$$  \hspace{1cm} (10)
where $E_a$, $v_a$ are elastic modulus and Poisson ratio of the soil, respectively, and $L$ is the soil length. For the Vlasov model, an additional soil length, typically taken as twice the beam length on both of its sides is assumed. This length was proven to be sufficient to capture the semi-infinite soil effect [20]. In the Pasternak model, this additional soil length is ignored.

To determine the parameter $\gamma$ an iterative method developed by Vallabhan and Das [15] is adopted as follows: First assume a value of $\gamma$ and calculate the $k_f$ and $k_m$ parameters form Eqs. (8) and (9), respectively. With these parameters computed, the new surface displacement $v(x)$ is evaluated. The process is repeated at each time step of the nonlinear solution algorithm until convergence is achieved within an acceptable tolerance.

2.2. Compatibility

The curvature at a section $x$ is related to the transverse displacements by:

$$\nu_{xx} - \chi = 0$$  \hspace{1cm} (11)

where $\nu$ is the vertical displacement of the beam, and $\chi$ is the curvature.

2.3. Material constitutive laws

The internal moment of the beam $M(x)$ is related to the curvature $\chi$ by a nonlinear constitutive relation

$$M(x) = \bar{g}(\chi(x))$$  \hspace{1cm} (12)

In this study the nonlinear relation in (12) is derived from a fiber discretization of the cross-section of the beam with nonlinear uniaxial stress–strain relations for the constituent materials. The two foundation forces $f_t$ and $m_t$ are related to their respective deformations by two other nonlinear relations as follows:

$$f_t = \bar{g}_f(\nu) \quad \& \quad m_t = \bar{g}_m(\nu)$$  \hspace{1cm} (13)

In the next sections, the strong form Eqs. (1)–(4), (5), (12) and (13) are solved for using the finite element method. Due to the nonlinear nature of Eqs. (12) and (13), a Newton-Raphson iteration strategy is used. The following discussion refers to a single Newton-Raphson iteration denoted by subscript $i$.

3. Displacement formulation of beam on two-parameter foundation

In a displacement formulation, the differential equations are solved based on a displacement field. Accordingly:

$$v(x) = a(x)V$$  \hspace{1cm} (14)

where $v(x)$ is the vertical displacement, and $a(x)$ is a matrix of $n_a$ shape functions, $n_a$ depends on the order of displacement shape functions, and $V$ is the vector of element end displacements.

The finite element formulation is considered by deriving the weighted integral of the equilibrium equation:

$$\int_{0}^{L} \delta v(x) [M_{xx} - m_{xx} + f_t - w] dx = 0$$  \hspace{1cm} (15)

where $\delta$ denotes derivation.

Integrating by parts twice the first term and once the third term, and ignoring the distributed load term $w$:

$$\int_{0}^{L} \delta v(x)^{i}_{xx} M dx + \int_{0}^{L} \delta v(x)^{i}_{xx} m dx + \int_{0}^{L} \delta v(x)^{i}_{tt} t dx + BT = 0$$  \hspace{1cm} (16)

where the Boundary term $BT$ equals to

$$BT = \delta vtM_{x_{10}} - \delta vuM_{t_{10}}$$

The consistent linearization of the nonlinear force-deformation relation for the beam and foundation yield:

$$M = k^{-1}_f \Delta v_{xx} + M^{-1}$$

$$m_t = k^{-1}_m \Delta v_{xx} + m^{-1}_t$$  \hspace{1cm} (17)

$$t_f = k^{-1}_f \Delta v_t + t^{-1}_f$$

Where $k$, $k_m$, $k_f$ are the derivatives of the nonlinear functions $g$, $g_m$, and $g_f$.

Substituting (17) into (16):

$$\int_{0}^{L} \delta v_{xx}(x) [k^{-1}_f \Delta v_{xx} + M^{-1}] dx + \int_{0}^{L} \delta v_{xx}(x) [k^{-1}_m \Delta v_{xx} + m^{-1}_t] dx$$

$$+ \int_{0}^{L} \delta v_t(x) [k^{-1}_f \Delta v_t + t^{-1}_f] dx = BT$$  \hspace{1cm} (18)

Substituting the predefined displacement shape functions $a(x)$ into (12), we get:

$$\delta v^T \left[ \int_{0}^{L} a_{xx}(x) k^{-1}_f a_{xx} dx + \int_{0}^{L} a_{xx}(x) k^{-1}_m a_{xx} dx + \int_{0}^{L} a_t(x) k^{-1}_f a_t dx \right] \Delta v =$$

$$\delta v^T \left[ P - \int_{0}^{L} a_{xx}(x) M^{-1} dx - \int_{0}^{L} a_t(x) m^{-1}_t dx - \int_{0}^{L} a_t(x) t^{-1}_f dx \right]$$  \hspace{1cm} (19)

From the arbitrariness of $\delta v$, we get:

$$\left( K_c^{-1} + K_m^{-1} + K_f^{-1} \right) \Delta v = P - M^{-1} - M_{xx}^{-1} - M_{tt}^{-1}$$  \hspace{1cm} (20)

where

$$K_c^{-1} = \int_{0}^{L} a_{xx}(x) k^{-1}_c a_{xx}(x) dx$$

is the beam element stiffness matrix,

$$K_m^{-1} = \int_{0}^{L} a_{xx}(x) k^{-1}_m a_{xx}(x) dx$$

is the Winkler foundation element stiffness matrix,

$$K_f^{-1} = \int_{0}^{L} a_t(x) k^{-1}_f a_t(x) dx$$

is the foundation force stiffness matrix,

$$M^{-1} = \int_{0}^{L} a_{xx}(x) M^{-1} dx$$

is the beam element resisting load vector,

$$M_{xx}^{-1} = \int_{0}^{L} a_{xx}(x) t^{-1}_f dx$$

is the Winkler foundation element resisting load vector,

$$M_{tt}^{-1} = \int_{0}^{L} a_t(x) t^{-1}_f dx$$

is the two-parameter foundation element resisting load vector, and $P$ is the vector of applied external loads.

4. Mixed formulation of beam on two-parameter foundation

In a two-field mixed formulation, the differential equations are solved based on both a displacement and a force field. For the foundation problem, it was proven earlier that this mixed approach is very advantageous from a numerical standpoint [26]. Accordingly, and similar to (14):

$$v(x) = a(x)V$$  \hspace{1cm} (21)

In addition $M(x) = b(x)M$  \hspace{1cm} (22)

where $M(x)$ is the bending moment, and $b(x)$ is a matrix of $n_a$ force interpolation functions, and $M$ is the vector of element end moments.

The finite element formulation is considered by deriving the weighted integral forms of both the compatibility and equilibrium equations:
The incremental section constitutive law is inverted substituted in (23). Accordingly:

\[
\int_0^L \delta M_i^T (x) \left[ \frac{\partial v(x)}{\partial x} - \chi \right] dx = 0
\]

(23)

\[
\int_0^L \delta M_i^T (x) \left[ \frac{\partial v(x)}{\partial x} - f^{-1} \Delta M_i - \chi \right] dx = 0
\]

(26)

\[
\text{where } f^{-1} \text{ is the section flexibility term at the previous Newton-Raphson iteration.}
\]

Substituting the predefined displacement shape functions and force interpolation functions into the weak form (26), we get:

\[
\begin{align*}
\delta M_i^T \left\{ \int_0^L b_i^T (x) a_{xx}(x) dx \right\} v - \int_0^L b_i^T (x) f^{-1} (x) b(x) dx \Delta M_i &= 0 \\
- \int_0^L b_i^T (x) \chi^{-1} (x) dx &= 0
\end{align*}
\]

(27)

From the arbitrariness of \( \delta M \), we get:

\[
\begin{align*}
\int_0^L b_i^T (x) a_{xx}(x) dx v - \int_0^L b_i^T (x) f^{-1} (x) b(x) dx \Delta M_i &= 0 \\
- \int_0^L b_i^T (x) \chi^{-1} (x) dx &= 0
\end{align*}
\]

(28)

Substituting \( v \) by \( v^{i-1} + \Delta v \), (28) becomes:

\[
\textbf{T} \Delta v^{i-1} = F^{i-1} \Delta M^{i-1} - v^{i-1} = 0
\]

(29)

where

\[
\textbf{T} = \int_0^L \sum_{i} b_i^T (x) a_{xx}(x) dx, \quad F^{i-1} = \int_0^L b_i^T (x) f^{-1} (x) b(x) dx.
\]

(30)

The order and continuity of stress and displacement interpolation functions are very important parameters in a mixed formulation. For stability of the formulation the rank of matrix \( \textbf{T} \) in the expression \( \textbf{T}^{-1} (\textbf{F}^{i-1})^{-1} \textbf{T} \) in Eq. (37) should not be larger than the rank of the flexibility matrix \( \textbf{F} \) for the limit case where the foundation stiffness matrix is zero. For this to be the case the number of unknowns \( n^e \) in vector \( v \) after excluding their rigid body modes should be less or equal to the number of unknowns \( n \) in vector \( \textbf{M} \):

\[
n^e \leq n
\]

(38)

While condition (38) is necessary for stability of the problem, there is no accuracy gain by increasing the order of the force field beyond that of the deformation field that respects the strain–displacement compatibility condition. The equality condition of (38), i.e. \( n^e = n \), is therefore the most efficient choice from a computational standpoint. As a result, the Babuska–Brezzi (B–B) stability conditions [28,29] for the beam on two-parameter foundation element states that the order of the displacement interpolation functions needs to be larger by two than that of the force interpolation functions.

5. Evaluation of model by numerical studies

5.1. Elastic beam on Vlasov foundation

The proposed two-parameter model with Pasternak and Vlasov effect (effect of the soil on either side of the beam is considered) was evaluated by analyzing a beam of finite length resting on an elastic foundation, which was first studied by Shirma and Ginger [13].

The beam length is 5 m, width \( b = 0.4 \) m and depth \( h = 1.0 \) m. The beam is made out of timber with an elastic modulus \( E_t = 10,500 \) MPa and Poisson ratio \( v_t = 0.25 \). The elastic foundation is sandy clay with an elastic modulus \( E_s = 45.4 \) MPa, Poisson ratio \( v_s = 0.25 \) and \( \gamma = 1.0 \). From Eqs. (8)–(10), the values of the
foundation parameters are: \( k_f = 3.081 \text{ MPa} \) and \( k_m = 12.449 \text{ kN} \).

The beam has free ends and is subjected to a concentrated moment of 50 kNm applied at the center as shown in Fig. 3. The beam is discretized into four mixed elements with cubic moment interpolation functions. Five integration points were assumed for each finite element. Since the problem is elastic, the load is applied at mid span under load control. To evaluate the effect of the semi-infinite foundation, the beam is analyzed using a Winkler, Pasternak, and Vlasov formulation. Fig. 4 shows the midspan moment rotation behavior of all models. From the figure, at the highest moment value of 50 kNm, the Winkler rotation was found to be 2.8 times larger than the Vlasov rotation. Fig. 5 shows the deflected shape of the models. Because of the added rotational resistance due to the Vlasov parameter, the end deflection of the Winkler foundation is about three times that of the Vlasov foundation. Furthermore, due to the effect of the soil on both sides of the beam in the Vlasov model, its end deflection is 2.1 times less than that of the Pasternak model. The bending moment is slightly underestimated if ignoring the Vlasov or Pasternak effects, as shown in Fig. 6.

The same beam with free ends is analyzed assuming the foundation to be tensionless, as shown in Fig. 7. An axial force \( P \) that equals 100 kN is applied under load control, while a moment \( M \) is applied incrementally under displacement control. Fig. 8 shows
the midspan moment rotation behavior of the beam for the Vlasov and Winkler foundations, respectively, using a mixed model with cubic moment interpolation functions. The Vlasov foundation moment resistance is found to be about 2.8 times larger than that of the Winkler foundation moment resistance at a rotation of 0.01 rad. Fig. 9 shows the foundation vertical displacement of both models at the ultimate load, which reveals that the Winkler model is having slightly larger deformations than the Vlasov model. From Fig. 10, which shows the foundation rotation for both models at the ultimate load, the Winkler model has a rotation 11% higher than that of the Vlasov model. The preceding discussion confirms the need to account for the semi-infinite soil effects in analyzing beam on foundation problems.

5.2. Inelastic beam on two-parameter foundation

The second numerical example represents an inelastic beam resting on a tensionless foundation. The main objective of this example is to compare the behavior of the Winkler one-parameter model, to the Pasternak and Vlasov two-parameter models. The beam is shown in Fig. 11, and has a length of 10 m, and a square cross-section with 100 mm dimension. For the Pasternak model, an additional soil length that equals twice the beam length on both of its sides was used. The adjacent soil effect on the beam depends on the soil modulus and depth of the soil layer. For the Vlasov model, an additional soil length of twice the foundation length (20 m) was added on both sides of the beam. The beam uniaxial stress–strain relation is elasto–plastic with Young's modulus $E = 200 \text{ GPa}$, yield strength of $207 \text{ MPa}$, and a hardening slope that equals 1.4%. The beam section is subdivided into 16 fibers. The underlying soil is 10 m Nevada Sand with properties as given by Pradhan and Desai [30] as follows: elastic modulus $E_s = 40.85 \text{ MPa}$, and Poisson ratio $\nu_s = 0.316$. The soil parameters are being calculated with the analytical method proposed by Vallabhan and Das [15] and described in Eqs. (6)–(10).

The loading condition consists of a transverse force and a moment acting at midspan, which is typical of foundation structures. The transverse force equals 70 kN, and is applied under load control, while the moment is applied incrementally under displacement control. The converged midspan moment rotation behavior of the beam is shown in Fig. 12 for the mixed model with 32
elements. In the figure, points \(A, A_0,\) and \(A_00\) represent yield points for the Vlasov, Pasternak and Winkler models, respectively. Similarly, points \(B, B_0\) and \(B_00\) represent the corresponding points at the ultimate state. The plot reveals that the stiffness is highest for the Vlasov model and is lowest for the Winkler model. The Vlasov model has also a higher yield and ultimate moment capacities, while the Winkler model has the lowest. The same plot is repeated in Fig. 13 using the displacement-based model. The distribution of the local parameters along the length of the beam, namely the bending moment, curvature, vertical displacement, rotation, and foundation forces are shown in Figs. 14–20 at the ultimate load stage. These figures reveal that the displacement model did not achieve convergence, even with 32 elements.
Fig. 14 shows the curvature distributions of the Winkler, Pasternak, and Vlasov models using the mixed formulation, while Fig. 15 shows the same distributions using the displacement formulation. The plots revealed that the mixed model was able to capture the curvature localization near the midspan and accurately predict the maximum curvature. The displacement model, however, failed to capture this behavior as it underestimated the maximum beam curvature value by a factor of 2.3. This is in part due to the assumed displacement shape functions, which cannot represent the steep curvature distribution accurately.

As observed in Figs. 16 and 17 the lift-off at the beam ends where the foundation force vanishes is severe for the Winkler model, and is much less for the Pasternak and Vlasov models. In addition, Fig. 17 reveals that the lift-off region is slightly higher for the Pasternak than for the Vlasov model, and that the foundation force is much smaller for the Pasternak than for the Vlasov model. Furthermore, from Fig. 16, due to consideration of the surrounding soil effect, the displacement at midspan is also much lower for the Vlasov and Pasternak models than for the Winkler model. The beam bending moment values along the length are higher for the Vlasov than for the Pasternak and Winkler models, as shown in Fig. 18, due to the additional moment resistance provided by the semi-infinite soil effects. The foundation rotation along the beam length is higher for the Winkler model than for the Pasternak and Vlasov models as shown in Fig. 19. The foundation moment resistance is zero for the Winkler model; while it is higher for the Vlasov model than for the Pasternak model, as shown in Fig. 20.

5.3. Shear wall foundation structure

Numerical analysis using the mixed model was conducted for the Aluminum shear wall foundation structure SSG04-06 tested by Gajan et al. [31] under an increasing lateral load. The footing is 2.8 m × 0.65 m, and has a Young's modulus of 70,000 MPa. The underlying soil is Nevada sand with modulus of elasticity 45 MPa and Poisson ratio 0.4. The yield force of the soil is 1238 kN/m, which is assumed to be at 35% of its bearing capacity. Fig. 21 shows the monotonic envelope of the moment–rotation plot at the middle of the foundation for both the Vlasov and Winkler models, as well as the experimental results. From the figure, it is observed that the Vlasov model is able to predict the behavior reasonably well, while the Winkler model under-predicts the moment capacity of the foundation. Fig. 22 shows the foundation force distribution at the ultimate load. The maximum foundation force equals 1504 kN/m, which exceeds the soil yield force, indicating that the soil has undergone inelastic deformations.

6. Conclusions

The paper presents a new inelastic element for the analysis of two-parameter beam on foundation problems. The element is derived from a two-field mixed formulation, where forces and deformations are approximated with independent interpolation functions. An iterative rational procedure to estimate the values of the two parameters of the foundation based on an assumption of plane strain for the soil medium was presented. This iterative behavior is conducted at each loading step of the nonlinear solution algorithm. The nonlinear response of structures resting on this newly developed two-parameter foundation model is analyzed following both a Vlasov and a Pasternak approach. Numerical examples to compare the behavior of the one-parameter and two-parameter models were conducted. The studies confirmed the importance of including the second parameter in estimating the foundation behavior, and revealed that accounting for the effect of the soil on both sides of the beam by adopting a Vlasov approach can substantially affect the nonlinear response. The studies also confirmed the superiority of the proposed mixed model in evaluating the inelastic complex behavior of these types of structures.
References


